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Complexity Analysis of Green Light Optimal Velocity Problem: An NP-complete Result for Binary Speed Choices

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ABSTRACT

The motion of ground vehicles in a signalized traffic can significantly affect the throughput and efficiency. This paper presents the complexity analysis of a known green light optimal velocity (GLOV) problem, *i.e.*, using the traffic light timing information to find optimal speed profiles that can avoid idling at red lights and minimize the trip time. A mathematical model for GLOV is formulated for the signalized traffic network, and then its problem complexity is analyzed by polynomial-time reducing a known NP-complete problem, *i.e.*, partition problem, into GLOV. The complexity analysis shows that GLOV with binary speed choices belongs to NP-complete, which means it cannot be numerically solved in polynomial time unless P=NP.

KEY WORDS- Traffic network; Green Light; Complexity Analysis; Optimal Speed.

1. INTRODUCTION

The explosive growth of vehicle number is causing increasing traffic congestions, especially in urban traffic network (Zhang *et al.*, 2011). It is reported that in US (2011) the traffic congestion in urban areas caused an estimated 56 billion pounds of additional CO_2 emission and 2.9 billion gallons of extra fuel, with a total cost of $121 billion (TTI, 2012). Adding additional infrastructure might mitigate traffic congestions, but definitely requires long time construction and huge demands for already limited land resources (Zhang *et al.*, 2011). One alternative approach is to optimize the usage of existing road transportation systems by cooperating vehicle motion control, and traffic signal management for the purpose of achieving a more efficient harmony traffic (Wen *et al.*, 2008; Lee *et al.*, 2013; Zheng *et al.*, 2015; Li *et al.*, 2012; Li *et al.*, 2013).

The signalized urban traffic network is attracting particular interest of researchers. Many studies have been conducted to improve the traffic efficiency at multiple intersections, which try to reduce the queue length and waiting time. This naturally leads to so-called green light optimal velocity (GLOV) problem. The GLOV problem is often formulated into a constrained optimal control framework, and then numerically solved by employing a rule-based or heuristic-based optimization algorithm. For instance, Mirchandani *et al.* (2001) proposed a traffic signal control algorithm to regulate the traffic flow by considering input detector data in a real-time way. Srinivasan *et al.* (2006) developed a distributed and unsupervised traffic responsive signal controller by using the multi-agent approach. A recent comprehensive review of traffic light control can be found in Li *et al.*, 2014. The aforementioned techniques are helpful to save travel time and fuel consumption. The vehicles are major components of road transportation, and the cooperative control of vehicle motions also significantly affects the urban traffic efficiency. For a series of consecutive intersections, the goal of cooperative control is to achieve so-called green wave traffic (Nagatani *et al.*, 2007; Corman *et al.*, 2009). For intersections with fixed traffic signals, the fulfillment of green wave traffic depends on how to adjust the speed profile of each vehicle, *i.e.*, speed planning. Recently, some studies have been conducted on speed planning at intersections with the aim of reducing trip time and/or improving fuel consumption. For example, Mandava *et al.* (2009) designed a speed planning algorithm at single intersection, which aimed to minimize the acceleration or deceleration rates, and to avoid stops at the red lights. Asadi and Vahidi (2011) developed a cruise control system, which used traffic light information to reduce idling time and fuel consumption. A predictive speed planning algorithm was proposed in Mahler *et al.* (2012), which focused on partial probabilistic information of traffic signals.

In short, most research now still focuses on limited number of intersections. Actually, the improvement of traffic efficiency relies on the implement of cooperative control of large scale traffic network. In such cases, the computation of GLOV becomes a challenging issue. To the best of our knowledge, this paper is the first study to analyze the complexity of large-scale GLOV. We have proved for the first time that GLOV with binary speed choices belongs to NP-complete, which means the problem cannot be solved in polynomial time unless P=NP. The complexity analysis is addressed by polynomial-time reducing a known NP-complete problem, *i.e.*, partition problem, into GLOV. The remainder of this paper is organized as follows: Section 2 formulates the mathematic model for GLOV. The complexity of GLOV is theoretically analyzed in Section 3. Section 4 includes some conclusion remarks.

2. Problem Statement

The green light optimal velocity (GLOV) problem is defined to find the optimal speed profiles that can avoid idling at red lights and minimize the trip time by using the traffic light information in an urban traffic network. The mathematic model is formulated for GLOV in this section.

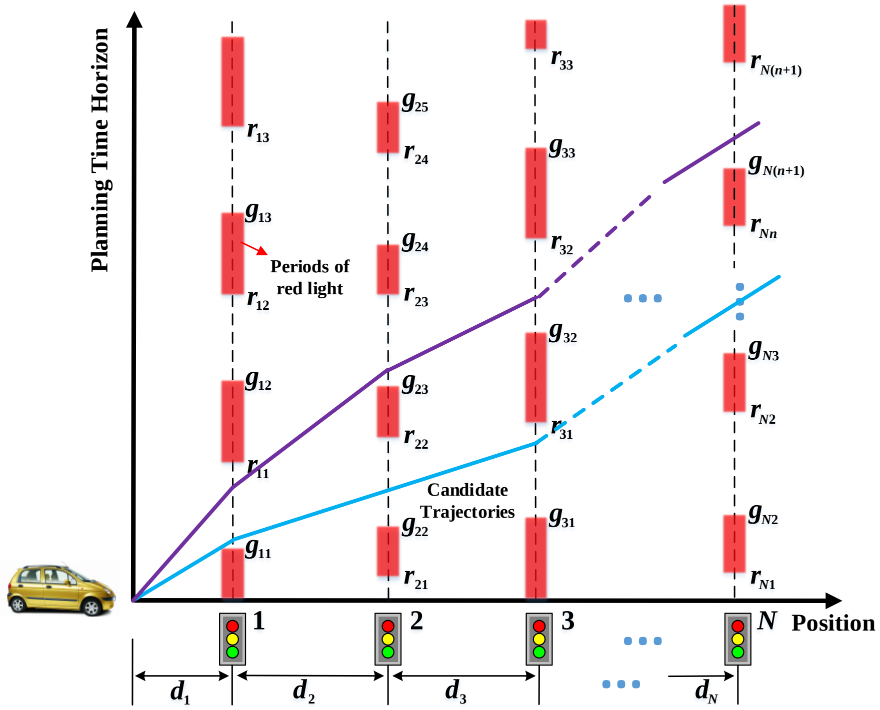

Fig. 1 Schematics map of traffic lights distributed over time-space in the route.

The route is assumed to have N intersections, as shown in Fig. 1 , with the parameters defined as follows: (1) A list of N intersections is denoted as $\mathbb{N} = \{1,2,\cdots,N\}$; (2) The length between consecutive intersections (called i-th segment) is represented as $d_i, i = 2,\cdots,N$, and d_1 is the distance to the first upcoming intersection; (3) The traffic light timing of each intersection is represented by triples $\{\mathrm{T}_i, G_i, g_{i1}\}$, where T_i denotes the phase cycle length; G_i denotes the green phase length, and $g_{i1}$is the start of the first green phase in i-th traffic light; (4) The initial speed is v_0 at the beginning time $t_0 = 0$. For simplicity, the green light phase of i-th traffic light is defined as $\bigcup_{j=1}^{\infty}[g_{ij}, r_{ij})$, where g_{ij} is the start of j-th green phase and r_{ij} is the start of j-th red phase. Note that the yellow phase is lumped into the red phase, *i.e.*,

$$\begin{cases} g_{ij} = g_{i1} + j \cdot T_i \\ r_{ij} = g_{i1} + j \cdot T_i + G_i \end{cases}. \tag{1}$$

It is assumed that the host vehicle can obtain the exact traffic light timing of each intersection

$\{T_i, G_i, g_{i1}\}$ by communication. The objective of GLOV is to minimize a performance index $J = f(V)$, *i.e.*, the trip time, without waiting at red lights by determining speed profiles $V = \{v_1, v_2, \cdots, v_N\}$ for each segment subject to certain constraints. The framework for GLOV is formulated as:

$$\min_{v_i} J = f(V)$$

subject to:
1) Time constraint;
2) Vehicle motion kinetics;
3) Allowable bounds for the speed;
4) Speed choices space; (2)

The four constraints stated in (2) are defined as follows.

(1) Time constraint, which ensures the arriving time $t_{\rm i}$ at *i*-th intersection lie in the green phase, *i.e.*,

$$t_i \in \bigcup_{j=1}^{\infty} [g_{ij}, r_{ij}), i \in \mathbb{N}. \tag{3}$$

(2) Vehicle motion kinetics, which assumes that the vehicle runs at constant speed in each segment ignoring the acceleration/deceleration process. Then, the arriving time t_i can be calculated as

$$t_i = \sum_{j=1}^{i} g_j(d_j, v_j) = \sum_{j=1}^{i} d_j/v_j, i \in \mathbb{N}, \tag{4}$$

where $g_j(d_j, v_j)$ represent the trip time in *i*-th segment.

(3) Allowable bounds for the speed, which is specified by the legal agency on each segment.

$$v_{i,\min} \le v_i \le v_{i,\max}, i \in \mathbb{N}. \tag{5}$$

(4) Speed choice space $\mathbb{S}$, which is required by designers, *e.g.*, the speed can be continuous, or limited to integer number of values, or even limited to binary values (*e.g.*, low speed and high speed).

$$v_i \in \mathbb{S}, i \in \mathbb{N}. \tag{6}$$

The GLOV is trip-time oriented, which means the objective is to find the optimal speed profiles in each segment such that the trip time is minimum without waiting at red lights. The performance index is written as:

$$J = \sum_{j=1}^{N} g_j(d_j, v_j) = \sum_{j=1}^{N} d_j/v_j. \tag{7}$$

In summary, the GLOV problem is formulated as

$$\min_{v_i} J = \sum_{j=1}^{N} d_j/v_j$$

subject to

$$\begin{cases} \sum_{j=1}^{i} d_j/v_j \in \bigcup_{j=1}^{\infty} [g_{ij}, r_{ij}) \\ v_{i,\min} \le v_i \le v_{i,\max} \\ v_i \in \mathbb{S} \end{cases}, i \in \mathbb{N}. \tag{8}$$

3. THE P/NP ANALYSIS

This section focuses the complexity analysis of GLOV defined in (8). It will be proved that GLOV defined in (8) has polynomial time solution if there is no speed limitation, whereas it becomes a NP-complete problem when there are binary speed choices in each segment. For sake of completeness, some definitions are stated here (Sipser *et al.*, 2006).

(1) Polynomial-time reduction. Problem $\mathcal{A}$ is said to be polynomial-time reduced to problem $\mathcal{B}$ if there is a polynomial-time algorithm for transforming inputs of problem $\mathcal{A}$ into inputs of problem $\mathcal{B}$, such that the transformed problem has the same output as the original problem.

(2) Class P. The general class of problems for which there exists some algorithm that can provide an answer in polynomial time is called "class P" (polynomial) or "P".

(3) Class NP. The general class of problems for which an answer can be verified in polynomial time is called "class NP" (Non-deterministic polynomial) or "NP".

(4) Class NP-complete. A set of problems to each of which any other NP problem can be reduced in polynomial time, and whose solution can be verified in polynomial time, is called NP-complete.

The problems that belong to NP-complete cannot be solved in polynomial time unless $\mathrm{P} = \mathrm{NP}$. Many computer scientists believe that $\mathrm{P} \neq \mathrm{NP}$, because up to now no one has been able to find a polynomial-time algorithm for any known NP-complete problems (Fortnow *et al.*, 2009). To prove a new problem to be NP-complete, we only need to polynomial-time reduce one known NP-complete problem into this new problem (Sipser *et al.*, 2006).

Lemma 1. (Kellerer *et al.*, 1997) Partition problem is NP-complete.

The partition problem is defined as: given n positive integers set $\{b_1, b_2, \cdots, b_n\}$, decide whether or not there exists a subset S_1 of $S = \{1,2,\cdots,n\}$ that satisfies

$$\sum_{j\in S_1} b_j = \sum_{j\notin S_1} b_j.$$

Theorem 1. Considering GLOV given by (8):

(1.1) Problem (8) belongs to P, if there is no speed limit and speed space $\mathbb{S}$ is continuous, *i.e.*, $v_{i,min} = 0, v_{i,max} = \infty$;

(1.2) Problem (8) belongs to NP-complete, if speed space $\mathbb{S}$ only have binary values *i.e.*, $\mathbb{S} = \{v_{low}, v_{high}\}$, $v_{i,\min} \le v_{low}, v_{high} \le v_{i,\max}, i \in \mathbb{N}$.

Proof: If there is no speed limit, the host vehicle can always pass the next intersection during the first green light phase. The optimal solution to the GLOV (8) can be constructed by the following steps:

Step 1.1: For the first intersection, if $g_{11} > 0$, then choose $v_1 = d_1/g_{11}$. Otherwise the following set intersection $[g_{11}, r_{11}] \cap (0, +\infty)$ must be not empty, and then $v_1$could be chosen as a very large speed, even be infinity mathematically;

Step 1.2: Suppose the vehicle arrive at the intersection $i-1$ at time $t_{i-1} = \sum_{k=1}^{i-1} d_k/v_k, i \in \mathbb{N}$. Checking the following set intersection:

$$[g_{ij} - t_{i-1}, r_{ij} - t_{i-1}] \cap (0, +\infty), \tag{9}$$

and choosing the first non-empty one. Then the speed in *i*-th segment is set as

$$v_i = \begin{cases} d_i/(g_{ij} - t_{i-1}) & if \quad g_{ij} - t_{i-1} > 0 \\ \infty & if \quad g_{ij} - t_{i-1} \le 0 \end{cases}.$$

Step 1.3: Repeat the process in step 1.2 until *i* reaches *N*.

The constructed solution is optimal to (8), because the trip time is always the minimum for the first *i* segments. Note that such strategy stated in Steps 1.1 to 1.3 may lead to infinity speed profiles mathematically.

To prove the Theorem 1.2, it is shown that partition problem can be polynomial-time reduced to GLOV (8). Given any instance of partition problem $\{b_1, b_2, \cdots, b_n\}$, GLOV is constructed as follows:

Step 2.1: Set b_i be the length of segment *i*, *i.e.*,

$$d_i = b_i, i \in \mathbb{N}.$$

Step 2.2: Make the green light phase of *i*-th intersection to be sufficient large, where $i = 1,2,\cdots,N-1,$ that host vehicle can pass these $N-1$ intersections by any means.

Step 2.3: Make the traffic light cycle of *N*-th intersection be sufficient large and only the following time point be green light phase

$$t_g = D\left(\frac{1}{2v_{low}} + \frac{1}{2v_{high}}\right), \tag{10}$$

where, $D = \sum_{i=1}^{N} d_i$ is the total length of *N* segments.

Obviously, the process of construction stated in Steps 2.1 to 2.3 can be finished in polynomial time. Then, we only need to prove that above GLOV is equivalent to the corresponding partition problem, *i.e.*, any solution to GLOV is a solution to partition problem, and vice versa.

Sufficiency analysis: Suppose there is one solution to partition problem, *i.e.*, there exists a subset S_1 of $S = \{1,2,\cdots,n\}$ which satisfies

$$\sum_{j\in S_1} b_j = \sum_{j\notin S_1} b_j.$$

Then make speed for subset S_1 be v_{low} and other segments be v_{high}. Hence, we have

$$t_N = \frac{\sum_{j\in S_1} b_j}{v_{low}} + \frac{\sum_{j\notin S_1} b_j}{v_{high}} = D\left(\frac{1}{2v_{low}} + \frac{1}{2v_{high}}\right) = t_g. \quad (11)$$

This means a solution to GLOV problem is obtained.

Necessity analysis: Suppose there is one solution to the GLOV problem, *i.e.*, there exists a subset $\tilde{S}_1$ of $S = \{1,2,\cdots,n\}$ and the advisory speed for subset $\tilde{S}_1$ is v_{low} and other segments is v_{high} such that (11) is satisfied. Note we have

$$\sum_{j\in \tilde{S}_1} b_j + \sum_{j\notin \tilde{S}_1} b_j = D. \quad (12)$$

Based on (11) and (12), we know

$$\sum_{j\in \tilde{S}_1} b_j = \sum_{j\notin \tilde{S}_1} b_j.$$

Therefore, a solution of partition problem is derived.

Hence, we claim that partition problem can be polynomial-time reduced to GLOV (8). According to Lemma 1, partition problem belongs to NP-complete. Thus, the GLOV (8) with binary speed choices also belongs to NP-complete. ■

Remark: If the speed space $\mathbb{S}$ is in integer domain, the complexity of GLOV (8) still remains an open question. Intuitively, it is much more difficult to solve GLOV (8) with more speed choices**.** For example, if the speed limit is set to be $v_{i,\min} = 1\text{m/s}, v_{i,\max} = 32\text{m/s}$, then there are 32 available speed choices for each segment. Then, the number of possible solutions (denoted as N_{ps}) to GLOV (8) is $N_{ps} = 32^N$. However, the number of possible solutions in Theorem 1.2 is only $N_{ps}^* = 2^N$.

4. CONCLUSIONS

This paper revisits the computation issue of the green light optimal velocity (GLOV) problem, which is to use the traffic light information to find the optimal speed profiles that can avoid idling at red lights and minimize the trip time. The complexity analysis is addressed for the first time by polynomial-time reducing a known NP-complete problem, *i.e.*, partition problem into GLOV. It is proved that GLOV has polynomial time solution if there is no speed limitation, whereas it becomes an NP-complete when there are only binary speed choices in each segment.

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